5.2. Bases for Eigenspaces

- Let A be an $n \times n$ matrix. The eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ of A are the roots of the polynomial $p_A(\lambda) = \det(A - \lambda I_n)$ (Theorem 5.9).
- For each eigenvalue λ_j of A, we have

$$E_{\lambda_j} = \{\vec{\mathbf{x}} \in \mathbb{R}^n \ : \ A\vec{\mathbf{x}} = \lambda_j \vec{\mathbf{x}}\}.$$

  This is the same as saying that

$$E_{\lambda_j} = \mathrm{NS}(A - \lambda_j I_n).$$

- Therefore, we can find a basis for each eigenspace of A using the technique for finding the basis of a null space described in Chapter 3.

Multiplicity

- Each eigenvalue λ_j of a matrix A has a multiplicity m_{λ_j} which is equal to the number of times the factor $(\lambda - \lambda_j)$ occurs in the factorization of $p_A(\lambda)$.
- We have $m_{\lambda_1} + m_{\lambda_2} + \cdots + m_{\lambda_k} = n$, because $p_A(\lambda)$ is a polynomial of degree n.
- Also, for each eigenvalue λ_j, we have $1 \leq \dim E_{\lambda_j} \leq m_{\lambda_j}$ (Theorem 5.17).
- Some of the λ_j may be complex numbers. So far, we've only dealt with finding eigenvectors for the real eigenvalues.

5.4. Multiplicity and Diagonalization

- If all of the eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ of A are real, then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A if and only if $\dim E_{\lambda_j} = m_{\lambda_j}$ for every eigenvalue λ_j. (Theorem 5.27)
- If this is the case, we say A is **diagonalizable over** $\mathbb{R}$.
- If $\mathcal{B}$ is the basis of $\mathbb{R}^n$ consisting of eigenvectors of A and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation defined by A, then $D = [f]_{\mathcal{B}}^{\mathcal{B}}$ is a diagonal matrix.
- Moreover, $D = P^{-1}AP$, where P is the $n \times n$ matrix whose columns are the eigenvectors in the basis $\mathcal{B}$. (Theorem 5.28)
- A matrix that has at least one non-real eigenvalue is not diagonalizable over $\mathbb{R}$. It may, however, be diagonalizable over $\mathbb{C}$, the set of complex numbers.

5.3. Complex Numbers

Theorem 5.19: Addition and multiplication in $\mathbb{C}$ satisfy the following properties:

1. Addition is commutative and associative in $\mathbb{C}$.
2. Multiplication is commutative and associative in $\mathbb{C}$.
3. Multiplication distributes over addition.
4. $0 \in \mathbb{C}$ is the additive identity and $1 \in \mathbb{C}$ is the multiplicative identity.
5. For all $\gamma = \alpha + \beta i \in \mathbb{C}$, the number $-\gamma = -\alpha - \beta i$ satisfies $\gamma + (-\gamma) = (-\gamma) + \gamma = 0$. ($-\gamma$ is the **additive inverse** of γ.)
6. For all nonzero $\gamma = \alpha + \beta i \in \mathbb{C}$, the number $\gamma^{-1} = \dfrac{\overline{\gamma}}{|\gamma|^2}$ satisfies $\gamma\gamma^{-1} = \gamma^{-1}\gamma = 1$. ($\gamma^{-1}$ is the **multiplicative inverse** of γ.)

The above properties show that $\mathbb{C}$ is a **field**.

5.4. Complex Diagonalization

- A real $n \times n$ matrix A not only defines a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$. A also defines a linear transformation $f_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$.
- If A has at least one complex eigenvalue, then $\mathbb{R}^n$ will not have a basis of eigenvectors for A. However, $\mathbb{C}^n$ might have such a basis.
- The matrix A will be diagonalizable over $\mathbb{C}$ if $\mathbb{C}^n$ has a basis of eigenvectors of A (Theorem 5.27).